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RECTANGULAR RIGID STAMP ON A NONLOCAL ELASTIC HALF-PLANE

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Abstract—First, the solution of the problem of a rectangular rigid stamp uniformly moving on an elastic half plane is given. Secondly, the solution is obtained for a motionless rectangular stamp on a nonlocal elastic half space. The finite distribution of the stress under the stamp has been compared with the unbounded stresses of local case. Copyright © 1996 Elsevier Science Ltd.

1. INTRODUCTION

The phenomenological interpretation of a medium as continuous is successful in general, but there exist some cases in which the inadequacy surfaces. In such cases the modelling using the continuous medium fails in representing the true picture of the actual situation by producing meaningless infinite stresses. The recently developed non-local theory (see Eringen (1987), Edelen (1976), Kunin (1982, 1983)) may be used to remove these inconsistencies and an example demonstrating the improvement is given below.

The problem of the motionless rectangular stamp on a linearly elastic half space has long been solved, (see Galin (1961), Muskhelishvili (1971), Artan (1994)) and the solution displays the unexplainable infinite stresses at the ends of the stamp. In this article the problem is remodelled using the nonlocal constitutive law for a linear elastic medium. But since the results for the classical case are needed for comparison, a slightly complicated form of the classical problem, the problem of a rectangular stamp uniformly moving on an elastic half plane, has first been solved. The comparison of the results of classical and nonlocal motionless rectangular stamps verifies once again the well-established fact: The nonlocal problem produces finite stress distribution under the stamp.

2. THE LOCAL PROBLEM

Let us consider a rectangular rigid stamp moving left with the constant speed w on an elastic half plane (see Fig. 1).

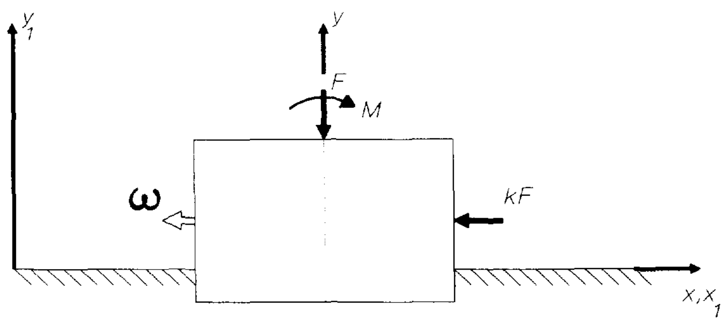

Fig. 1. Rectangular rigid stamp on an elastic half plane.

Denote by u_1, v_1 the respective displacement components of the elastic half-plane in x_1, y_1 coordinate system and let the Lamé moduli be λ, μ of the linearly elastic isotropic half-plane. Then the Navier's equations of motion are

$$(\lambda+\mu)\frac{\partial}{\partial x_1}\left(\frac{\partial u}{\partial x_1}+\frac{\partial v}{\partial y_1}\right)+\mu\Delta u-\frac{\rho\,\partial^2 u}{\partial t^2}=0 \tag{1a}$$

$$(\lambda+\mu)\frac{\partial}{\partial y_1}\left(\frac{\partial u}{\partial x_1}+\frac{\partial v}{\partial y_1}\right)+\mu\Delta v-\frac{\rho\,\partial^2 v}{\partial t^2}=0 \tag{1b}$$

where ρ is the mass density. We define a new variable $\theta^*(x_1, y_1, t)$ by the following relations

$$u=-\frac{\lambda+\mu}{\mu}\frac{\partial^2\theta^*}{\partial x_1\,\partial y_1};\quad v=\frac{\lambda+2\mu}{\mu}\frac{\partial^2\theta^*}{\partial x_1^2}+\frac{\partial^2\theta^*}{\partial y_1^2}-\frac{\rho}{\mu}\frac{\partial^2\theta^*}{\partial t^2} \tag{2}$$

Then (1a) is satisfied identically. On the other hand we obtain

$$\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial y_1^2}-\frac{\rho}{\lambda+2\mu}\frac{\partial^2}{\partial t^2}\right)\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial y_1^2}-\frac{\rho}{\mu}\frac{\partial^2}{\partial t^2}\right)\theta^*(x, y, t)=0 \tag{3}$$

from (1b). Let us denote the longitidunal and transversal wave speeds by

$$c_1=\sqrt{\frac{\lambda+2\mu}{\rho}};\quad c_2=\sqrt{\frac{\mu}{\rho}} \tag{4}$$

respectively. Now we choose a second coordinate system x, y which is constrained to move with the stamp and hence the coordinate transformation is given by

$$x=x_1-wt;\quad y=y_1 \tag{5}$$

On the other hand we define a function $\theta(x, y)$ by

$$\theta(x,y)=\theta^*(x_1, y_1, t)=\theta[x_1-wt, y] \tag{6}$$

then (3) becomes

$$\left[\left(1-\frac{w^2}{c_1^2}\right)\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right]\left[\left(1-\frac{w^2}{c_2^2}\right)\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right]\theta(x,y)=0 \tag{7}$$

The displacement components (2) are

$$u(x,y)=-\frac{\lambda+\mu}{\mu}\frac{\partial^2\theta}{\partial x\,\partial y};\quad v(x,y)=\left(\frac{\lambda+2\mu}{\mu}-\frac{w^2}{c_2^2}\right)\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial y^2} \tag{8}$$

in terms of $\theta(x, y)$. Using the Hooke constitutive relations

$$\sigma_x=\lambda\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+2\mu\frac{\partial u}{\partial x};\quad \sigma_y=\lambda\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+2\mu\frac{\partial v}{\partial y};\quad \tau_{xy}=\mu\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right) \tag{9}$$

and the following notation

$$a = -\frac{\lambda+\mu}{\mu}; \quad b = \frac{\lambda+2\mu}{\mu} - \frac{w^2}{c_2^2}; \quad c = -(\lambda+2\mu) - \lambda\frac{w^2}{c_2^2}; \quad d = \lambda$$

$$e = (3\lambda+4\mu)-(\lambda+2\mu)\frac{w^2}{c_2^2}; \quad g = \lambda+2\mu; \quad l = \lambda+2\mu-\mu\frac{w^2}{c_2^2}; \quad h = -\lambda \tag{10}$$

we can write

$$u(x,y) = a\frac{\partial^2\theta}{\partial x\,\partial y}; \quad v(x,y) = b\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial y^2}; \quad \sigma_x = c\frac{\partial^3\theta}{\partial x^2\,\partial y}+d\frac{\partial^3\theta}{\partial y^3}$$

$$\sigma_y = e\frac{\partial^3\theta}{\partial x^2\,\partial y}+g\frac{\partial^3\theta}{\partial y^3}; \quad \tau_{xy} = l\frac{\partial^3\theta}{\partial x^3}+h\frac{\partial^3\theta}{\partial x\,\partial y^2} \tag{11}$$

We define the differential operators L_1, L_2 by

$$L_1 = \left(1-\frac{w^2}{c_1^2}\right)\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}; \quad L_2 = \left(1-\frac{w^2}{c_2^2}\right)\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2} \tag{12}$$

which are commutative; that is $L_1L_2 = L_2L_1$. Hence, $L_1L_2\theta = 0$ whenever $\theta = \theta_1+\theta_2$ and $L_1\theta_1 = 0$; $L_2\theta_2 = 0$. If $w < c_2 < c_1$, then both L_1 and L_2 are elliptic operators. Equation (7) can be written in the form

$$\left(\frac{\partial^2}{\partial x^2}+\frac{1}{k_1^2}\frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2}{\partial x^2}+\frac{1}{k_2^2}\frac{\partial^2}{\partial y^2}\right)\theta(x,y) = 0 \tag{13}$$

by the notation

$$k_1^2 = 1-\frac{w^2}{c_1^2}; \quad k_2^2 = 1-\frac{w^2}{c_2^2}. \tag{14}$$

Operators L_1, L_2 take the simple forms

$$L_1 = \left(\frac{\partial^2}{\partial x^2}+\frac{1}{k_1^2}\frac{\partial^2}{\partial y^2}\right) = 4\frac{\partial^2}{\partial z_1\,\partial \bar{z}_1}; \quad L_2 = \left(\frac{\partial^2}{\partial x^2}+\frac{1}{k_2^2}\frac{\partial^2}{\partial y^2}\right) = 4\frac{\partial^2}{\partial z_2\,\partial \bar{z}_2} \tag{15}$$

if we use the complex variables z_1, z_2 defined by

$$z_1 = x+ik_1y; \quad z_2 = x+ik_2y \tag{16}$$

Hence (13) becomes

$$\frac{\partial^4\theta}{\partial z_1\,\partial \bar{z}_1\,\partial z_2\,\partial \bar{z}_2} = 0 \tag{17}$$

At the end of some simple but lengthy calculations (see Artan (1994)) the problem of the determination of normal stresses under the stamp reduces to the task of finding the solution of the following singular integral equation with Cauchy-type kernel

$$\pi p f'(x) = \pi k q \sigma_y(\xi, y)_{y=0} + \int_{-\infty}^{+\infty} \frac{\sigma_y(\xi, y)_{y=0}}{\xi - x} \mathrm{d}\xi \tag{18}$$

where

$$p = -\frac{2E}{(1+\nu)m^2}\left(1-\frac{1-2\nu}{2-2\nu}m^2\right)^{-1/2}\left[(1-\tfrac{1}{2}m^2)^2-\left(1-\frac{1-2\nu}{2-2\nu}m^2\right)^{1/2}(1-m^2)^{1/2}\right] \tag{19}$$

$$\lim_{m\to 0} p = \frac{E}{2(1-\nu^2)} \tag{20}$$

$$q = \frac{\left(1-\frac{1-2\nu}{2-2\nu}m^2\right)^{-1/2}\left[1-\frac{1}{2}m^2-\left(1-\frac{1-2\nu}{2-2\nu}m^2\right)^{1/2}(1-m^2)^{1/2}\right]}{-\frac{1}{2}m^2}; \tag{21}$$

$$\lim_{m\to 0} q = -\frac{1-2\nu}{2-2\nu} \tag{22}$$

$$m = \frac{w}{c_2} \tag{23}$$

and k is the coefficient of friction. The solution of (18) is

$$(\sigma_y)_{y=0} = \frac{kqpf'(x)}{1+k^2q^2} - \frac{X_0(x)\mathrm{e}^{G(x)}}{(1+k^2q^2)\pi^2}\int_{-\infty}^{+\infty}\frac{\pi p f'(t)\mathrm{e}^{-G(t)}}{X_0(t)(t-x)}\mathrm{d}t + \frac{X_0(x)R(x)\mathrm{e}^{G(x)}}{\pi\sqrt{1+k^2q^2}} \tag{24}$$

where

$$\mathrm{e}^{G(x)} = \left(\frac{l-x}{l+x}\right)^{\delta}; \quad \tan(\pi\delta) = \frac{-1}{kq}; \quad 0 \leqslant \delta \leqslant \tfrac{1}{2} \tag{25}$$

$R(x)$ is a single valued function, $X_0(x)$ is a rational function.

Both ends of the rectangular stamp are in contact with the half plane. Therefore the stresses at both ends are unbounded. The choice

$$X_0(x) = \frac{1}{l-x} \tag{26}$$

is appropriate. The other unknown function reduces to a constant whose value is to be found from the overall vertical equilibrium equation. Since the stamp is rectangular and is kept horizontal by applying the external force in an appropriate position, the derivative of the shape function of the stamp is zero:

$$f'(x) = 0$$

Substitution of this simplification into eqn (24) gives

$$\phi(x) = \frac{Q_0\left(\dfrac{l-x}{l+x}\right)^{\delta}}{(l-x)\pi\sqrt{1+k^2q^2}} \tag{27}$$

The equation of vertical equilibrium is

$$\int_{-l}^{+l} \phi(x)\,\mathrm{d}x = -F \tag{28}$$

Using this equation the constant Q_0 is found as

$$Q_0 = -F\sqrt{1+k^2q^2}\sin\pi\delta \tag{29}$$

Therefore the stresses under the stamp are

$$\phi(x) = -\frac{F}{\pi}\sin\pi\delta\left(\frac{l-x}{l+x}\right)^{\delta}\frac{1}{l-x} \tag{30}$$

3. THE BASIC EQUATIONS OF NONLOCAL ELASTICITY

The governing equations of the nonlocal theory of elasticity are (Eringen (1974), Eringen (1976), Eringen and Balta (1981), Altan (1989))

$$t_{kl,k} = 0 \tag{31}$$

$$t_{kl} = \int_V \tau'_{kl}(x, x')\,\mathrm{d}x' \tag{32}$$

$$\tau'_{kl} = \lambda'(|x'-x|)e'_{rr}(x')\delta_{kl} + 2\mu'(|x'-x|)e'_{kl}(x') \tag{33}$$

$$e'_{kl} = \tfrac{1}{2}(u'_{k,l}+u'_{l,k})u'_k \equiv u_k(x') \tag{34}$$

In these equations the comma as a subscript denotes the partial derivative, that is:

$$t_{kl,m} = \frac{\partial t_{kl}}{\partial x_m}, \quad u'_{k,l} = \frac{\partial u'_k}{\partial x'_l} \tag{35}$$

On the other hand, there exists addition on repeated indices. Equations (31) and (34) are the same both in local and nonlocal elasticities. Equation (32) expresses the fact that the stress at an arbitrary point x depends on the strains at all the points x' of the body. λ' and μ' are Lamé constants of the nonlocal medium and they depend on the distance between x and x'. They can be taken as (see Eringen (1974))

$$\lambda' = \alpha(|x'-x|)\lambda \tag{36}$$

$$\mu' = \alpha(|x'-x|)\mu \tag{37}$$

where λ and μ are the Lamé constants of the local case. $\alpha(|x'-x|)$ is called the kernel function and is the measure of the effect of the strain at x' on the stress at x. The kernel satisfies the following property (see Eringen (1976))

$$\int_{V} \alpha(|x'-x|)\,\mathrm{d}x' = 1 \tag{38}$$

Substituting (36) and (37) in (33)

$$\tau'_{kl} = \alpha(|x'-x|)\{\lambda e'_{rr}(x')\delta_{kl}+2\mu e'_{kl}(x')\} \tag{39}$$

is obtained. (32) can be written in the suggestive form

$$t_{kl} = \int_{V} \alpha(|x'-x|)\sigma_{kl}(x')\,\mathrm{d}x' \tag{40}$$

using the notation

$$\sigma_{kl}(x') = \sigma'_{kl} = \lambda e'_{rr}(x')\delta_{kl}+2\mu e'_{kl}(x') \tag{41}$$

then the equilibrium eqn (31) becomes

$$t_{kl,k} = \int_{V} \alpha_{,k}(|x'-x|)\sigma_{kl}(x')\,\mathrm{d}x' = 0 \tag{42}$$

Substituting the identity

$$\alpha_{,k}\sigma'_{kl} = -\alpha_{,k'}\sigma'_{kl} = -(\alpha\sigma'_{kl})_{,k'}+\alpha\sigma'_{kl,k'} \tag{43}$$

and using the divergence theorem

$$-\int_{\partial V} \alpha\sigma'_{kl}n_{k'}\,\mathrm{d}x'+\int_{V} \alpha\sigma'_{kl,k'}\,\mathrm{d}x' = 0 \tag{44}$$

is obtained. When the body extends to infinity in all directions or the surface tractions are in equilibrium then the surface integral in (44) vanishes and we obtain

$$\int_{V} \alpha\sigma'_{kl,k'}\,\mathrm{d}x' = 0 \tag{45}$$

If $\sigma'_{kl,k'}$ is a continuous function of x' then

$$\sigma_{kl,k} = 0 \tag{46}$$

hence the satisfaction of the governing eqn (31) gives the result (46) which implies that the nonlocal stresses can be calculated using (40).

4. THE SOLUTION FOR THE NONLOCAL STAMP

In this article, the kernel function will be chosen as

$$\alpha(|x'-x|) = \begin{cases} B\left\{1-\dfrac{|x'-x|}{a}\right\} & |x'-x| < a \\ 0 & |x'-x| > a \end{cases} \tag{47}$$

where B is a constant and is determined using (38) as

$$B = \frac{1}{a} \tag{48}$$

On the other hand a is the atomic distance which will be taken as

$$a = 0.00000004\,\text{cm} \tag{49}$$

The elasticity modulus, Poisson ratio and l will be chosen as

$$E = 100000\,\text{kg/cm}^2, \quad \nu = 0.3, \quad l = 20\,\text{cm} \tag{50}$$

We calculate the stresses first for the absence of friction. Hence $k = 0$ and using (20), (22) and (25)

$$\delta = \tfrac{1}{2}, \quad p = 54945.05\,\text{kg/cm}^2, \quad q = -0.2857$$

is found. Substitution of (30) into (40) gives

$$t(x) = -\frac{F}{\pi a}\left\{\int_{x-a}^{x+a} \frac{1}{\sqrt{l^2-x'^2}}\,\mathrm{d}x' - \int_{x-a}^{x} \frac{x-x'}{a}\frac{1}{\sqrt{l^2-x'^2}}\,\mathrm{d}x' - \int_{x}^{x+a} \frac{x'-x}{a}\frac{1}{\sqrt{l^2-x'^2}}\,\mathrm{d}x'\right\} \tag{51}$$

The evaluation of integrals gives the stresses under the stamp in the form

$$t(x) = -\frac{F}{\pi}\left\{\frac{(a+x)\arcsin\left(\frac{a+x}{l}\right)}{a^2} + \frac{(a-x)\arcsin\left(\frac{a-x}{l}\right)}{a^2} - \frac{(2x)\arcsin\left(\frac{x}{l}\right)}{a^2} + \frac{\sqrt{-a^2-2ax+l^2-x^2}+\sqrt{-a^2+2ax+l^2-x^2}-2\sqrt{l^2-x^2}}{a^2}\right\} \tag{52}$$

In this expression, the stresses of the local case (30) are obtained in the limit for a approaching zero. In the case of frictional stamp (for $k = 0.6165$)

$$p = 54945.05\,\text{kg/cm}^2, \quad q = -0.2857, \quad \delta = \tfrac{4}{9}$$

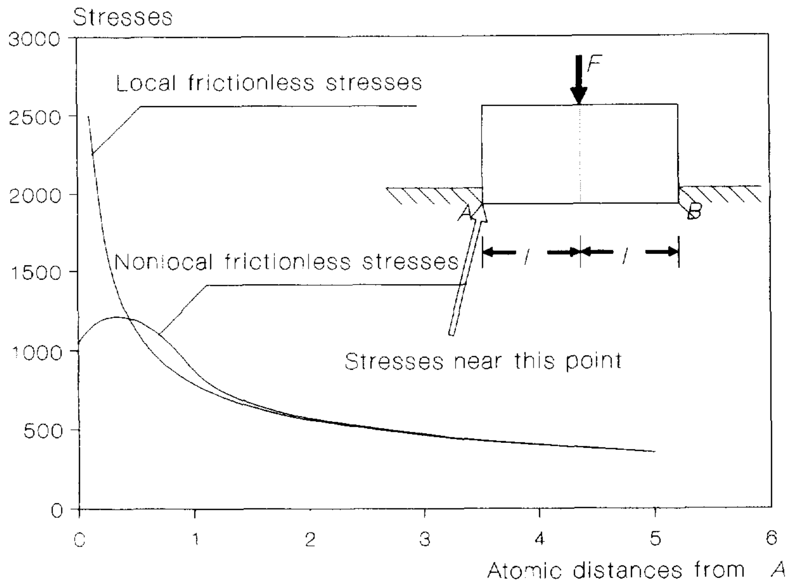

Fig. 2. Stresses in the local and nonlocal frictionless cases at the front end.

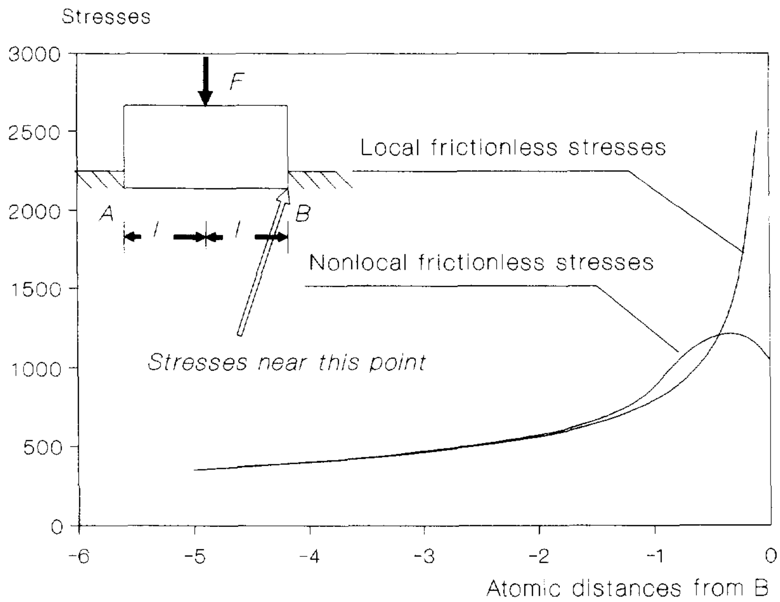

Fig. 3. Stresses in the local and nonlocal frictionless cases at the rear end.

is obtained using (20), (22) and (25). Substitution of (30) in (40) gives the stresses under the stamp in the form

$$t(x) = \frac{-F\sin \pi\delta}{\pi a}\int_{x-a}^{x+a}\left(1-\frac{|x-x'|}{a}\right)\left(\frac{l-x'}{l+x'}\right)^{\delta}\frac{1}{l-x'}\mathrm{d}x' \tag{53}$$

The integrals in this expression cannot be evaluated in closed form. Therefore the stresses under the frictional stamp have been calculated numerically. In Figs 2–7 the stresses under the stamp are given by diagrams.

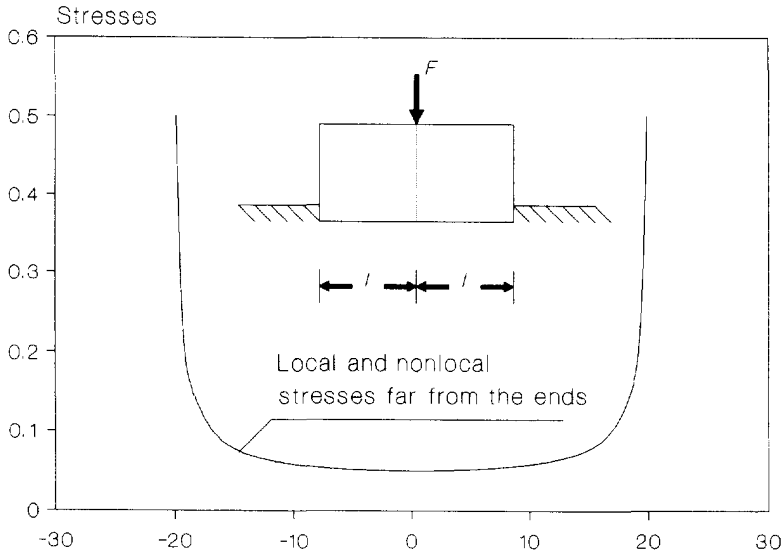

Fig. 4. Local and nonlocal frictionless stresses far from the ends.

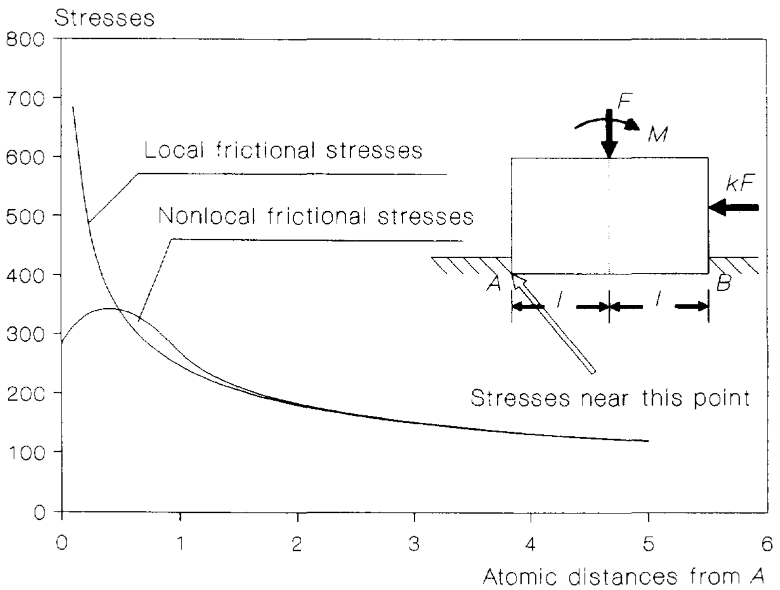

Fig. 5. Local and nonlocal frictional stresses at the front end.

5. CONCLUSION

The stress singularities occuring in the classical solution of the problem is an impediment for an assessment of the true situation and results are hardly usable for design purposes. But here the investigation and comparison of both frictional and frictionless cases show that the nonlocal stresses are finite even at the points where local stresses are infinite. In several problems this advantage of the nonlocal theory exhibits itself (see Eringen (1976), Eringen and Balta (1981)). If in any problem of elasticity the solution of the local problem has no singularity both formulations give exactly the same results.

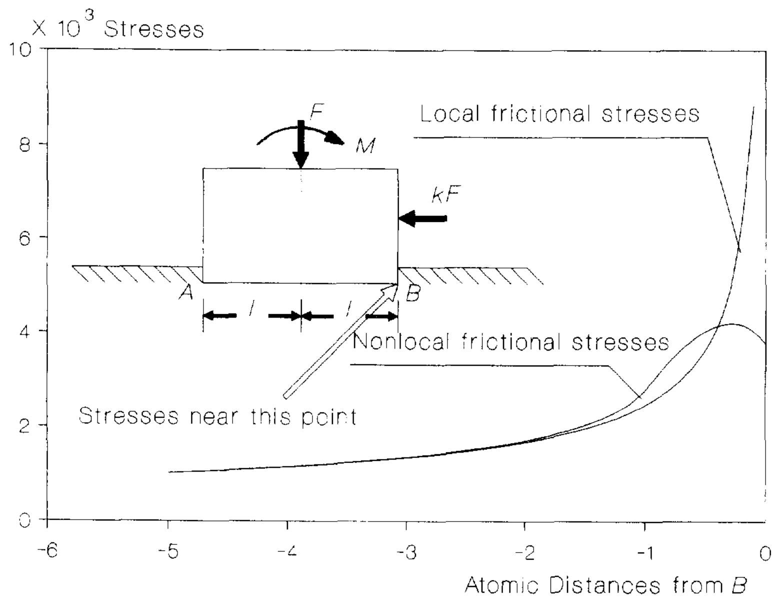

Fig. 6. Local and nonlocal frictional stresses at the rear end.

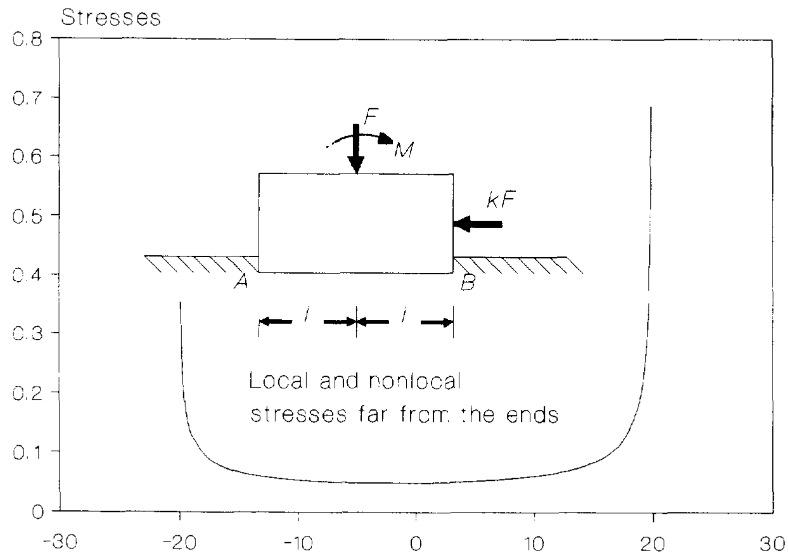

Fig. 7. Local and nonlocal frictional stresses far from the ends.

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